Indian Statistical Institute Bangalore
Statistics and Mathematics Unit
M. Math II Year

Course: Several Complex Variables

Back-paper Exam, Date: 04 June, 2025

Time: 3 hours | Maximum marks: 50

All questions are compulsory.

NOTATIONS: (1) $\mathbb{R}$: set of all real numbers, (2) $\mathbb{C}$: set of all complex numbers, (3) $\mathbb{C}^n$: the n-dimensional complex number space, (4) $\mathbb{R}^n$: the n-dimensional real number space, (5) For a given $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$ and $\mathbf{r} = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n$ with $r_i > 0$ for all $1 \leq i \leq n$,

$$\mathbb{D}^n(\mathbf{a}, \mathbf{r}) := \{\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_i - a_i| < r_i \text{ for all } 1 \leq i \leq n\}.$$

For $n = 1, a \in \mathbb{C}$ and a positive $r \in \mathbb{R}$, we will write $\mathbb{D}(a, r)$, instead of $\mathbb{D}^1(\mathbf{a}, \mathbf{r})$.

---

1. (a) Show that the unit bidisc $\mathbb{D}^2(\mathbf{0}, \mathbf{1}) = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$ and the unit ball $\mathbb{B}^2(\mathbf{0}, \mathbf{1}) := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$ are not biholomorphically equivalent.

   (10 marks)

   (b) Define $f : \mathbb{C}^2 \to \mathbb{C}^2$ by

   $$f(z_1, z_2) = (z_1, z_2 + g(z_1)),$$

   where $g : \mathbb{C} \to \mathbb{C}$ is an entire function with $g(0) = g'(0) = 0$. Prove that f maps $\mathbb{C}^2$ biholomorphically onto $\mathbb{C}^2$, $f(0, 0) = (0, 0)$ and the Jacobian $J_f((0, 0))$ is the 2×2 identity matrix.

   (10 marks)

2. (a) Carry out a coordinate transformation in $\mathbb{C}^3$ in order to make $f : \mathbb{C}^3 \to \mathbb{C}$, defined by

   $$f(z) = z_1 z_2 z_3$$

   vanish of order 3 relative to the new third coordinate at the origin.

   (10 marks)

   (b) Let $\Omega \subset \mathbb{C}^n$ be a connected domain, $f : \Omega \to \mathbb{C}$ be a holomorphic function and there is a nonempty open set $U \subset \Omega$ such that $f\big|_U = 0$. Show that $f \equiv 0$ on Ω. **(10 marks)**

3. (a) Let $\Omega \subset \mathbb{C}^n$ be a domain and $K \subset \Omega$ be nonempty and compact. Give the definition of holomorphically convex hull of K relative to Ω.

   (2 marks)

   (b) Show that if $\Omega \subset \mathbb{C}$ is a domain containing $\overline{\mathbb{D}(0, 1)}$, then the holomorphically convex hull of $\{z \in \mathbb{C} : |z| = 1\}$ relative to Ω is $\overline{\mathbb{D}(0, 1)}$.

   (5 marks)

   (c) Describe the Cartan-Thullen characterizations of domains of holomorphy.

   (3 marks)

Best wishes!